THE VARIATION OF THE INDEX OF REFRACTION IN A VORTEX TUBE

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Three models of a vortex tube (isentropic, isothermal, and isochoric) are considered as optical inhomogeneities. Expressions relating the index of refraction to the coordinate are obtained. The possibility of obtaining Schlieren pictures of such flows is discussed.

In experimental work involving vortex flow it is sometimes possible to use shadow or interference methods, by means of which one can measure the distribution of the index of refraction in the flow field and use that distribution to calculate the distributions of other variables [1].

A vortex is associated with a pressure change, which is a function of the particle velocity [2]. The pressure change is accompanied by a change of density, and the latter is uniquely related to the index of refraction. Consequently, a vortex is associated with a field of varying index of refraction and can, in principle, produce a shadow or interference image.

Under different conditions there appear vortices of different types, depending on the thermodynamic process which takes place. In this work we shall consider three models of a vortex tube: isentropic, isothermal, and isochoric.

Consider a vertical vortex tube in an ideal fluid, with a core of radius r_0 inside which the fluid is in solid-body rotation

$$v = \omega r. \tag{1}$$

The core induces in the surrounding fluid a flow with velocity

$$v = \omega r_0^2/r. \tag{2}$$

Introducing the dimensionless radius $\bar{r} = r/r_0$ and dimensionless velocity $\bar{v} = v/\omega r_0$, we can rewrite (1) and (2) in the form

$$\bar{v} = \bar{r} \quad \text{for} \quad r \leqslant r_0, \tag{3}$$

$$\bar{v} = \frac{1}{\bar{r}} \quad \text{for} \quad r \geqslant r_0, \tag{4}$$

which clearly shows the dependence of $\bar{v}$ on $\bar{r}$.

Inside the core, where the flow is rotational, the pressure and the density are related to the velocity by Euler's equation of motion

$$\frac{v^2}{r} = \frac{1}{\rho}\,\frac{dp}{dr}. \tag{5}$$

Outside the core, where the flow is potential, we can use Bernoulli's equation

$$\frac{v^2}{2} + \int \frac{dp}{\rho} = \text{const}. \tag{6}$$

Introducing the dimensionless pressure $\bar{p} = p/p_\infty$ and the dimensionless density $\bar{\rho} = \rho/\rho_\infty$, and using (1)–(4), we reduce Euler's and Bernoulli's equations to the form

$$k\bar{v}^2 - \frac{1}{\varkappa}\int \frac{d\bar{p}}{\bar{\rho}} = \text{const}, \tag{7}$$

$$k\bar{v}^2 + \frac{1}{\varkappa}\int \frac{d\bar{p}}{\bar{\rho}} = \text{const}. \tag{8}$$

In both cases

$$k = \omega^2 r_0^2/2a_\infty^2, \quad a_\infty^2 = \varkappa p_\infty/\rho_\infty.$$

We shall now consider the three models of vortex flow assuming that the above velocity field exists in all three cases, and shall calculate the pressure, density, and temperature fields.

Consider, for example, an isentropic vortex. Using the dimensionless variables introduced above, we represent the Poisson equation

$$p/p_\infty = (\rho/\rho_\infty)^\varkappa$$

in the dimensionless form

$$\bar{p} = \bar{\rho}^{\varkappa}. \tag{9}$$

Substituting (9) in (7) and (8), we obtain

$$k\bar{v}^2 - \frac{1}{\varkappa - 1}\bar{\rho}^{\varkappa-1} = \text{const} \tag{10}$$

for $r \leq r_0$, and

$$k\bar{v}^2 + \frac{1}{\varkappa - 1}\bar{\rho}^{\varkappa-1} = \text{const} \tag{11}$$

for $r \geq r_0$. The value of the constant in (10) is determined by the boundary conditions at the core boundary ($\bar{v} = 1$, $\bar{\rho} = \bar{\rho}_0$),

$$k\bar{v}^2 - \frac{1}{\varkappa - 1}\bar{\rho}^{\varkappa-1} = k - \frac{1}{\varkappa - 1}\bar{\rho}_0^{\varkappa-1}. \tag{12}$$

To determine the constant in (11), we use the conditions at infinity, $\bar{v} = 1/\bar{r} = 0$, $\bar{\rho} = 1$. This yields

$$k\bar{v}^2 + \frac{1}{\varkappa - 1}\bar{\rho}^{\varkappa-1} = \frac{1}{\varkappa - 1}. \tag{13}$$

Equation (13) represents the relation between the dimensionless density and dimensionless velocity

The Variation of $\bar{\rho}$, $\bar{p}$, and $\bar{T}$ with $\bar{r}$ for Various Vortex Models

Inside the core ($\bar{v}=\bar{r}$)	Outside the core ($\bar{v}=1/\bar{r}$)
Isentropic vortex	
$\bar{\rho}=[1+k(\varkappa-1)(\bar{r}^{2}-2)]^{1/(\varkappa-1)}$	$\bar{\rho}=[1-k(\varkappa-1)/\bar{r}^{2}]^{1/(\varkappa-1)}$
$\bar{p}=[1+k(\varkappa-1)(\bar{r}^{2}-2)]^{\varkappa/(\varkappa-1)}$	$\bar{p}=[1-k(\varkappa-1)/\bar{r}^{2}]^{\varkappa/(\varkappa-1)}$
$\bar{T}=1+k(\varkappa-1)(\bar{r}^{2}-2)$	$\bar{T}=1-k(\varkappa-1)\bar{r}^{-2}$
Isothermal vortex	
$\bar{\rho}=\exp[k\varkappa(\bar{r}^{2}-2)]$	$\bar{\rho}=\exp(-k\varkappa/\bar{r}^{2})$
$\bar{p}=\exp[k\varkappa(\bar{r}^{2}-2)]$	$\bar{p}=\exp(-k\varkappa/\bar{r}^{2})$
$\bar{T}=1$	$\bar{T}=1$
Isochoric vortex	
$\bar{\rho}=1$	$\bar{\rho}=1$
$\bar{p}=1+k\varkappa(\bar{r}^{2}-2)$	$\bar{p}=1-k\varkappa/\bar{r}^{2}$
$\bar{T}=1+k\varkappa(\bar{r}^{2}-2)$	$\bar{T}=1-k\varkappa/\bar{r}^{2}$

(or coordinate) outside the vortex core. At the core boundary equation (13) becomes

$$\frac{1}{\varkappa-1}\bar{\rho}_0^{\varkappa-1}=\frac{1}{\varkappa-1}-k.$$

Using this expression, we can eliminate the term with $\bar{\rho}_0$ from (12), and thus obtain a relation between the dimensionless density and the dimensionless velocity (or coordinate) inside the vortex core:

$$\bar{\rho}^{\varkappa-1}=1+k(\varkappa-1)(\bar{v}^{2}-2).$$

In an isentropic process the pressure and the temperature are related as

$$p/p_\infty=(T/T_\infty)^{\varkappa/(\varkappa-1)},$$

or, in dimensionless form,

$$\bar{p}=\bar{T}^{\varkappa/(\varkappa-1)}. \tag{14}$$

Taking account of this relation and of equation (9), we can obtain the distribution of the dimensionless pressure or temperature as a function of $\bar{v}$ or $\bar{r}$.

In the case of the isothermal or isochoric vortices, instead of the Poisson equation we use the Clapeyron equation

$$p=\rho RT,$$

or, in dimensionless form,

$$\bar{p}=\bar{\rho}\bar{T}.$$

In the isothermal case $\bar{T}=1$ and

$$\bar{p}=\bar{\rho}, \tag{15}$$

whereas in the isochoric case $\bar{\rho}=1$ and

$$\bar{p}=\bar{T}. \tag{16}$$

Proceeding as before, we can use (15) or (16) to derive all the necessary relations from (7), (8). The results are collected in the table, together with the results for the isentropic vortex.

Now, using the expressions for the density as a function of the coordinate, one can easily derive the expression for the index of refraction as a function of the coordinate. The connecting relation is the Gladstone-Dale equation [3]

$$\rho/\rho_\infty=(n-1)/(n_\infty-1).$$

Denoting the right side by $\bar{n}$, we rewrite this equation in the form

$$\bar{\rho}=\bar{n}.$$

Now it is clear that the variation of the quantity $(n-1)/(n_\infty-1)$ is exactly the same as that of the dimensionless density, i.e., in the isochoric case $\bar{n}$ is constant everywhere, in the isentropic case it is

$$\bar{n}=\begin{cases}[1+k(\varkappa-1)(\bar{r}^{2}-2)]^{1/(\varkappa-1)} & \text{for } \bar{r}\leqslant 1,\\ \left[1-k(\varkappa-1)\dfrac{1}{\bar{r}^{2}}\right]^{1/(\varkappa-1)} & \text{for } \bar{r}>1\end{cases} \tag{17}$$

and in the isothermal case it is

$$\bar{n}=\begin{cases}\exp[k\varkappa(\bar{r}^{2}-2)] & \text{for } \bar{r}\leqslant 1,\\ \exp\left[-k\varkappa\dfrac{1}{\bar{r}^{2}}\right] & \text{for } \bar{r}\geqslant 1\end{cases}$$

The index of refraction reaches its minimum value at the center of the vortex. Denoting this value by $\bar{n}_c$ ($\bar{n}=\bar{n}_c$ at $\bar{r}=0$), we find in the isentropic case

$$\bar{n}_c=[1-2k(\varkappa-1)]^{1/(\varkappa-1)}, \tag{18}$$

and in the isothermal case

$$\bar{n}_c=\exp(-2k\varkappa). \tag{19}$$

Thus, $\bar{n}$ varies between 1 and the value given by (18) or (19).

Clearly, an optical instrument can "see" such a two-dimensional vortex only if the difference $1-\bar{n}_c$ is sufficiently large, i.e., larger than the sensitivity threshold of the instrument. In the isentropic case

$$1-\bar{n}_c = 1-[1-2k(\varkappa-1)]^{1/(\varkappa-1)} \tag{20}$$

and in the isothermal case

$$1-\bar{n}_c = 1-\exp(-2k\varkappa). \tag{21}$$

From (20) and (21) it is clear that these differences are determined by the values of $\varkappa$ and $k = \omega^2 r_0^2/2\varkappa RT_\infty$. Consequently, we can construct graphs of $1-\bar{n}_c$ as a function of $\varkappa$, T_∞, ωr_0. Clearly, the difference $1-\bar{n}_c$ depends on $\varkappa$ only in the isentropic case. This dependence is shown in Fig. 1a.

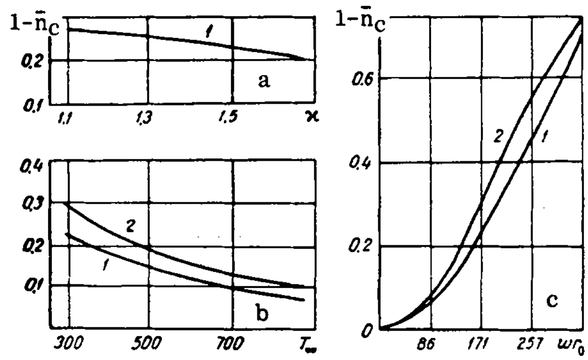

Fig. 1. $1-\bar{n}_c$ as a function of (a) $\varkappa$, (b) T_∞, (c) ωr_0. 1) isentropic vortex; 2) isothermal vortex..

It appears from the graphs in Fig. 1 (these were constructed for the case $a = 342.4$ m/sec, $\varkappa = 1.4$, $\omega r_0 = 0.5a$) that it is easier to obtain a shadow or interference image of the vortex in the case of a cold gas with low ratio of specific heats and high velocity at the core boundary. Also, an isothermal vortex will be easier to visualize than an isentropic vortex.

The above formulas allow us to explore the possibility of using optical methods to visualize vortex flows under various experimental conditions.

Assume, for example, that we observe a section of a horizontal rectilinear isentropic vortex tube, bounded by the two normal sections A and B. Let the origin of the coordinate system lie in A, and let the z axis coincide with the tube axis, which is horizontal. We throw a parallel beam of light in the z direction.

The section under consideration constitutes an optical inhomogeneity with a cylindrical field of the index of refraction. The values n, dn/dx, and dn/dy are constant along straight lines parallel to the z axis [1]. The theory shows that in this case the deflections ε_x and ε_y of the light beam in the x and y directions, respectively, are given by

$$\frac{\partial \varepsilon_x}{\partial z} = \frac{1}{n_\infty}\frac{\partial n}{\partial x},$$
$$\frac{\partial \varepsilon_y}{\partial z} = \frac{1}{n_\infty}\frac{\partial n}{\partial y}.$$

In our case

$$\frac{\partial n}{\partial x} = \frac{\partial n}{\partial r}\frac{x}{r}, \quad \frac{\partial n}{\partial y} = \frac{\partial n}{\partial r}\frac{y}{r},$$

and, since $\partial n/\partial r$, x/r, and y/r are independent of z, we have

$$\varepsilon_x = \frac{1}{n_\infty}\frac{\partial n}{\partial r}\frac{x}{r}\Delta z, \quad \varepsilon_y = \frac{1}{n_\infty}\frac{\partial n}{\partial r}\frac{y}{r}\Delta z.$$

The deflection of the beam in the radial direction is

$$\varepsilon = \sqrt{\varepsilon_x^2+\varepsilon_y^2} = \frac{\Delta z}{n_\infty}\frac{\partial n}{\partial r}, \tag{22}$$

where Δz is the distance between the sections A and B. This deflection increases with increasing Δz and $\partial n/\partial r$.

Using (17), we have for the case of the isentropic vortex

$$\frac{\partial n}{\partial r} =$$
$$= \frac{2k(\bar{n}_\infty-1)\bar{r}}{r_0}[1+k(\varkappa-1)(\bar{r}^2-2)]^{(2-\varkappa)/(\varkappa-1)} \quad \text{for } \bar{r}<1$$

and

$$\frac{\partial n}{\partial r} =$$
$$= \frac{2k(n_\infty-1)}{r_0\bar{r}^3}\left[1-k(\varkappa-1)\frac{1}{\bar{r}^2}\right]^{(2-\varkappa)/(\varkappa-1)} \quad \text{for } \bar{r}>1.$$

At the core center, as well as at infinity, $\partial n/\partial r = 0$. At the core boundary ($\bar{r} = 1$) the value of $\partial n/\partial r$ is, in both cases,

$$\left(\frac{\partial n}{\partial r}\right)_0 = \frac{2k(n_\infty-1)}{r_0}[1-k(\varkappa-1)]^{(2-\varkappa)/(\varkappa-1)}. \tag{23}$$

It can be easily seen that (23) represents the maximum value of $\partial n/\partial r$. Substituting this in (22), we obtain the maximum value of the radial deflection of the beam

$$\varepsilon_0 = \frac{2k(n_\infty-1)\Delta z}{r_0 n_\infty}[1-k(\varkappa-1)]^{(2-\varkappa)/(\varkappa-1)}.$$

For $n_\infty = 1.000272$, $\omega r_0 = 0.5a_\infty$, $\omega = 1000$ cycles, $a = 342$ m/sec and $\Delta z = 0.01$ m we obtain

$$\varepsilon_0 = 3.5\cdot 10^{-4}\ \text{rad},$$

which is not too far from the sensitivity threshold of a well-adjusted "IAB" instrument. The longer the section of the vortex tube, the easier it would be to observe the vortex by optical methods. However, the basic factor which governs the variation of the index of refraction in the vortex, is the quantity ωr_0. When this quantity is too small as compared with the speed of sound, the instrument will be unable to register the flow. Therefore, the use of optical methods for the measurement of vortices would make sense only in high-speed gas flows.

NOTATION

a—speed of sound; n—index of refraction; p—pressure; r—radius; T—absolute temperature; v—linear velocity; ε—angular deflection of the beam; $\varkappa$—adiabatic exponent; ρ—density; ω—angular velocity; $\bar{v}$, $\bar{p}$ etc.—dimensionless parameters; v_0, ρ_0—values at the core boundary; p_∞, ρ_∞ etc.—stagnation values; n_c—index of refraction at center of vortex.

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